Personal Notes on

Complex Analysis

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Abstract

This note is based on MAT389: Complex Analysis, complex notes can be found here

Contents

1 Complex Numbers

1.1 Basic Definitions

Lets start with a fundamental definition

Definition: A *complex number* is a number of the form

$$z = x + iy \quad \text{where x,y} \in \mathbb{R}$$

and i satisfies $(i)^2 = -1$ The set of all complex numbers is denoted $\mathbb{C}$.

- We can also extract separate information from $z \in \mathbb{C}$

  $$Re(z) = x$$

  and

  $$Im(z) = y$$

- The modulus of z is

  $$|z| = \sqrt{x^2 + y^2}$$

- The complex conjugate of z is

  $$z^* = \overline{z} = x - iy$$

Looking into the fundamental rules of complex number of we have some basic algebra

- Add/subtraction

  $$(a + ib) + (c + id) = (a + c) + i(b + d)$$

- Multiplication:

  $$(a + ib)(x + iy) = ax + iay + ibx - by$$

  Note that if we have

  $$z \cdot \overline{z} = |z|^2$$

- Division: if z is a non-zero complex number, then

  $$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

  Then lets say another complex number w

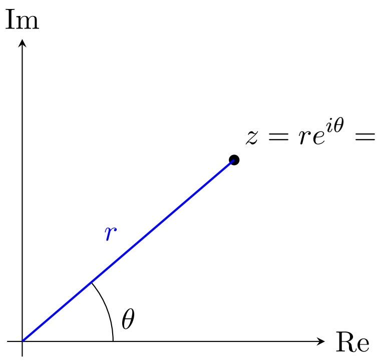
  $$\frac{w}{z} = \frac{w \cdot \overline{z}}{|z|^2}$$

There are also a few Corollares to remember

$$z \cdot \overline{z} = |z|^2 \tag{1}$$
$$|\overline{z}| = |z| \tag{2}$$
$$|z \cdot w| = |z| \cdot |w| \tag{3}$$
$$\overline{zw} = \overline{z} \cdot \overline{w} \tag{4}$$

Also all the usual properties of Algebra with $\mathbb{R}$ continue to hold(commutative, distributive etc)

Next, a complex number can always be represented in polar representation



Doing algebra in complex polar notation is also simpler

Example 1: if z = $|z|e^{i\theta}$, w = $|w|e^{i\phi}$
then by doing the multiplication it is simply an addition of phase angle

$$z \cdot w = |z||w|e^{i(\theta+\phi)}$$

What about for more complex numbers, we would need De Moivres Theorem

Theorem: The DeMoivre's Theorem explains that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

But how defined is θ we would need something to show it

Definition: Argument:
we define an argument of z $z \in \mathbb{C}$ to be any θ such that

$$z = |z|e^{i\theta}$$

θ is only unique up to addition of integer multiples of 2π.
We define the principal value to be the argument $\theta \in [-\pi, \pi]$, and define

$$\arg(z) = \theta \in [-\pi, \pi]$$

1.2 Vector Calculations

We can do vector calculation in $\mathbb{R}^2$ using complex analysis.

1. Vector addition = Complex addition
2. dot product = $Re(z\overline{w})$

This lead to the consequences that the modulus follows the trangle inequality

$$|z + w| =\leq |z| + |w|$$

1.3 Roots of Complex Numbers

Consider the equation $X^n = a$, how many solutions do this equation has? If X is a real variable, then the number of solution to this equation is

1. no solutions
2. 1 solution or more

3. n Solutions

However, over $\mathbb{C}$ the equation will always have n distinct solutions.

Example 2:

$$x^n = -1$$

Let x = $\cos\theta + i\sin\theta Arg(-1) = -\pi$, by Demoivre theorem, we have that

$$n\theta = -\pi + 2\pi k$$

therefore, there are n distinct solutions

$$\theta_n = -\pi/n + 2\pi k/n$$

1.4 Subsets of the Plane

- The open disc of radius R centered at $z_0 \in \mathbb{C}$ is the set

$$\{z \in \mathbb{C} | |z - z_0| < R\}$$

- if $D \subseteq \mathbb{C}$ is a subset and $w_0 \in D$ then w_0 is an interior of D. If it is an open disc, centered at w_0 contained in D.
- A set $D \subseteq \mathbb{C}$ is open if every point of D is an interior point(boundary point is not an interior point).
- if $D \subseteq \mathbb{C}$ then the boundary of D(∂D is the set of all points "on the edge of D."
- w is a boundary point if every open disc centered at w includes points that are included and not included in D.
- D is open if and only if it contains no boundary points
- C is closed if and only if its complement

$$D = C^c = \{z \in \mathbb{C} | z \notin C\}$$

  is open.
- There might be sets that are neither open nor closed, like $D = \mathbb{C}$ is both open and closed.

Here are some notions of point-set topology(yay topology):

- A set $D \subseteq \mathbb{C}$ is *connected* if for any p,q $\in D$ there is a curve joining p to q, lying entirely in D.
- A *Domain* $D \subseteq \mathbb{C}$ is a **non-empty, open connected** set.
- **The Point at Infinity:** Idea for $z \in \mathbb{C}$ st $|z| \neq 0$ then $w = \frac{1}{z} \in \mathbb{C}/\{0\}$, then we can add w = 0 that corresponds to $z = \infty$. For some larger number M, we can have

$$\{|z| > M\} \leftrightarrow \{|w| < \frac{1}{M}\}$$

  Geometrically, how we can think of this point of infinity is that, consider a R^2 plane, the spherical projection allows the plane to project to a sphere each with a unique point. However at the north pole of the sphere, the tangent lines goes to infinity. Therefore, if we have a set in R^2 that describes the plane, adding a point of infinity essentially turns it into a sphere.

1.5 Complex Functions

A function of a complex variable $z \in \mathbb{C}$ is a rule of assigning one complex number to another complex number.

$$z \in D \to f(z) \in \mathbb{C}$$

where D is the domain of f. $f(D)$ is the range of f. (note that the domain here is the 'classical' domain which doesn't have information about open and connected etc.) Here comes another definition needed

- **limits:** let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Eg $z_n = 1 + (\frac{1}{n})i \quad or \quad z_n = n + i\cos(n)$ we say $\lim_{n\to\infty} z_n = A$ if z_n approaches A as $n \to \infty$. However this is not rigorous enough.
  Rigorously: $\forall \epsilon > 0 \quad \exists N$ st. $\forall n \geq N$, $|z_n - A| < \epsilon$
- **Properties:** if $\{z_n\}, \{w_w\}$ is a sequeces, then

$$\lim_{n\to\infty} z_n = A, \lim_{n\to\infty} w_n = B$$

  then

  1. $\lim_{n\to\infty}(z_n + \lambda w_n) = A + \lambda B$
  2. $\lim_{n\to\infty} z_n w_n = AB$
  3. if $B \neq, then \quad \lim_{n\to\infty} \frac{z_n}{w_n} = \frac{A}{B}$

Now we can talk about continuity

Definition: if $D \subseteq \mathbb{C}, f : D \to \mathbb{C}, z_0 \in \overline{D} = D \cup \partial D$. Then say

$$\lim_{z\to z_0} f(z) = L$$

if *for any sequence*$\{z_n\} \subseteq D$, we have $\lim_{n\to\infty} z_n = z_0$, therefore

$$\lim_{n\to\infty} f(z_n) = L$$

Say that $\lim_{z\to\infty} f(z) = L$ if $\forall \epsilon > 0$, $\exists M$ st.

$$|f(z) - L| < \epsilon$$

on the set $\{|z| > M\} \cap D$, which is the formal definition of "The limit at Infinity."

Example 3: $f(z) = e^{-|z|}$ has $\lim_{z\to\infty} f(z) = 0$

Therefore, with all the knowledge we can finally come out with a definition of continuity

Definition: A function $f : D \to \mathbb{C}$ is continuous at $z_0 \in D$ if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Say f is continuous on D, if it is continuous at every point of D. Let f, g be functions continuous at z_0. Then:
(i) If $\lambda \in \mathbb{C}$, then $f + \lambda g$ is continuous at z_0.
(ii) $f \cdot g$ is continuous at z_0.
(iii) If $g(z_0) \neq 0$, then $\frac{f}{g}$ is continuous at z_0.
for instance we know that f(z) = z is continuous, which implies that z^k with $k = 1, 2, 3, 4...$ is also continuous, which we can also prove that any polynomials of any orders of z is also continuous.

2 Infinite Series & Exponentials

suppose we have a sequence of complex numbers $z_1, z_2....$ we define the n0th partial sum to be

$$S_n = \sum_{j=1}^{n} z_j = z_1 + z_2... + z_n$$

we say that $\sum_{j=1}^{\infty} z_j$ converges and $\sum_{j=1}^{\infty} z_j = S$ if

$$\lim_{n\to\infty} S_n = S \in \mathbb{C}$$

if $\lim_{n\to\infty} S_n$ does not exist, we say the sum diverges. If we write in complex number then

$$\sum_{j=1}^{n} z_j = (\sum_{j=1}^{n} x_j) + i(\sum_{j=1}^{n} y_j)$$

so the summability of z_j is the same to the summability of x_j, y_j which are **Real**. However, how do we test if something is converging, we have the **Test of Convergence**

Theorem: If $\sum_{j=1}^{\infty} |z_j|$ converges, then so does $\sum_{j=1}^{\infty} z_j$

one of the test is the **Ratio Test** such that if

$$\lim_{j\to\infty} \frac{|z_{j+1}|}{|z_j|} = a < 1$$

then the series converges. Rest of the content about exponential please refer to Lecture three of the course note, there is no need to reinvent the wheel lol. link just need to add up a bit more detail, on **Page 8** of the note, the proof becomes

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \sum_{k=2m} \frac{(iy)^k}{k!} + \sum_{k=2m+1} \frac{(iy)^{k+1}}{(k+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (y)^{2m}}{2m!} + i \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m+1}}{(2m+1)!} = \cos(y) + i\sin(y)$$

also on **Page 9** of the note, we can define

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

the properties of $e^a e^b = e^{a+b}$ originates from the uniqueness of the properties of the functions, such that there is only one function that satisfies the properties, and any other functions of lets say $e^{a+b} = g$ proves that they are the same.

2.1 Logarithm

the formal definition of logarithm is

$$Logz = log|z| + iArg(z)$$

where $Arg(z) \in [-\pi, \pi)$ is the principal value that is called the **Principal Branch of Log**. However, there are many branches. lets say we draw any ray through the origin $D = \{t(\cos\theta + i\sin\theta) | t \geq 0, t \in \mathcal{R}\}$, take a point on D z_0 that associates with an angle θ_0 we may now define the logarithm of any point using this branch cut as

$$logz = log|z| + i\phi$$

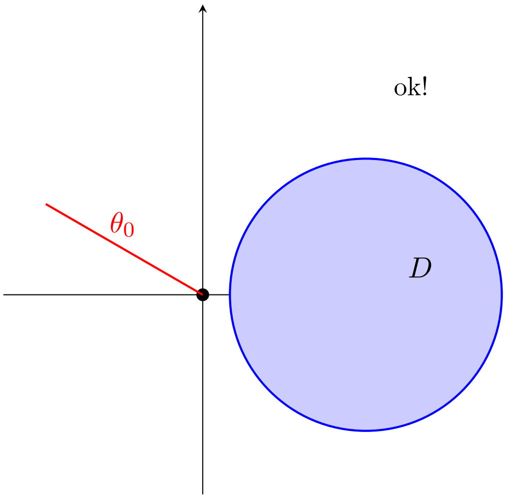
$\phi \in [\theta_0, \theta_0 + 2\pi)$ is the argument of z.

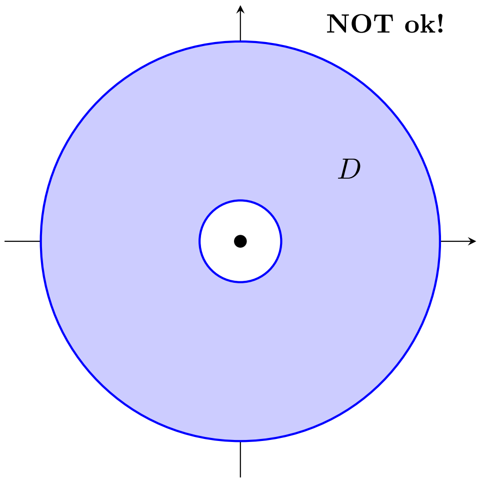
Example 4: Solve $z^{1+i} = 4$

$$\begin{aligned} z^{1+i} &= e^{(1+i)logz} \\ \rightarrow logz &= \frac{1-i}{2}(2log2 + 2\pi ki) \\ &= logz + \pi k + i(\pi - log2) \\ z &= 2(-1)^k e^{\pi k}(\cos(log2) - i\sin(log2)) \end{aligned}$$

Note that an important definition is **Single Valued Branch** such that

- For every $z \in D$ f(z) is one of the possible values of the multi-valued expression.
- f is continuous throughout D.
- No contradictions occur when moving around any closed path in D; i.e. if you return to the same point z, the function value f(z) also returns to its original values.





This is because if we take a loop from $\theta = 0 \rightarrow 2\pi$, then at the same position we have the log

$$log(z(2\pi)) = \ln 1 + i(2\pi) = 2\pi i \neq \log(z(0))$$

this means that we cannot make log z continuous on any region containing a full loop around 0 such that at the same position there are infinite many values. If a domain has a single valued branch of log z then log z is differentiable and

$$\frac{d}{dz}\log z = \frac{1}{z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{x - iy}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}$$

3 Line Intengrals

lets first introduce a definition of a parametrized curve

Definition: A **Parametrized Curve** is a continuous map $\gamma(t) = x(t) + iy(t) \in \mathbb{C}$ with $a \leq t \leq b$, $\gamma : [a, b] \to \mathbb{C}$

- $\gamma(t)$ is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ for $a \leq t_1 < t_2 < b$
- $\gamma(t)$ is closed if $\gamma(a) = \gamma(b)$
- A **Parametrized curve** is C^1(continuously differentiable) if $\gamma'(t) = x'(t) + iy'(t)$ exists for all $t \in [a, b]$ and x',y' are continuous on $[a, b]$
- if g = u + iv is complex valued function, γ is a piecewise C' parametrized curve then

$$\int_\gamma g = \int_a^b g(\gamma(t))\gamma'(t)dt = \int_a^b (ux' - vy' + i(vx' + uy'))dt$$

recall that the length of a parametrized curve is defined by

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)|dt$$

following the triangle inequality, we can obtain the relation that

$$|\int_\gamma g| \leq \int_a^b |g||\gamma'(t)|dt \leq \max(|g(z)|) \cdot \text{length}(\gamma)$$

with that in mind, the Green's Theorem reads

Definition: Green Theorem:
for $\Omega \subseteq \mathbb{C}$ domain(connected open set) st $\partial\Omega$ is a finite collection of piece wise C' simple closed curves orient $\partial\Omega$ st. Ω lies to the left as we walk along $\partial\Omega$ (say $\partial\Omega$ is positively oriented)

now lets define a function $f(z) = p(z) + iq(z)$ that is differentiable, $p, q \in \mathbb{C} \to \mathbb{R}$ Then

$$\boxed{\int_{\partial\Omega} f dz = i \int_\Omega (\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})dxdy}$$

How do we understand the relation in Green's Theorem. We check equality of the real and imaginary parts. On the **Real Part**

$$\gamma(t) = x(t) + iy(t)$$

$$\int_{\partial\Omega} (p + iq)(x' + iy')dt = \int_{\partial\Omega} f dz$$

which can be proven by expanding the terms, and similarly for the imaginary part. First of all

$$\int_{\partial\Omega} pdx - qdy = \int_\Omega (-\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dxdy$$

and for the imaginary part

$$\int_{\partial\Omega} qdy - pdx = \int_\Omega (\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x})dxdy$$

Example 5: Let Ω be a domain, $\partial\Omega = \gamma$, a simple, closed, positively oriented piecewise C^1 curve, if $p \notin \gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - p} = \begin{cases} 1 & \text{if p} \in \Omega \\ 0 & \text{if } p \notin \Omega \end{cases}$$

lets say for case $p \notin \Omega$, then $f = \dfrac{1}{z-p}$ is $C^1 in \Omega$ so Green's Theorem applies (recall that z = x + iy)

$$\frac{\partial f}{\partial x} = \frac{-1}{(z-p)^2}, \quad \frac{\partial f}{\partial y} = \frac{-i}{(z-p)^2}$$

so

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

For case 1:$p \in \Omega$ the green's theorem doesn't apply since the fraction is not C^1 in Ω. Now lets define a smaller set Ω_ϵ, basically Ω with a hole in the center that contains the point p. the boundary line integral becomes

$$\int_{\partial\Omega} f dz = \int_{\partial\Omega} f dz - \int_{\partial D_\epsilon} f dz + \int_{\partial D_\epsilon} f dz$$

where $D_\epsilon(p) = \{z \in \mathbb{C} || z-p| < \epsilon\}$ and $\partial D_\epsilon(p)$ is positively oriented. Note that

$$\int_{\partial\Omega} f dz - \int_{\partial D_\epsilon} f dz = \int_{\partial\Omega_\epsilon} f dz = 0 \quad \text{for case 2}$$

and now we can compute $\displaystyle\int_{\partial D_\epsilon(p)} f dz$ with the dircht parametrize curve of the boundary. Let

$$\partial D_\epsilon(p) = p + \epsilon e^{it}$$

then

$$\int_{\partial D_\epsilon(p)} f dz = \int_0^{2\pi} \frac{1}{\epsilon e^{it}} i\epsilon e^{it} dt = 2\pi i$$

4 Analytic Functions

Definition: A complex function f(z) defined for $z \in D$ is **differentiable** at $z_0 \in D$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z+h) - f(z_0)}{h}$$

exists

- If f is differentiable at z, for all $z \in D$ then f is said to be **analytic** on D
- if f is **analytic** on all of $\mathbb{C}$ then f is said to be **entire**

Some examples is here

Example 6: We show that

$$\frac{d}{dz} e^z = e^z.$$

By definition of the derivative,

$$\frac{d}{dz} e^z \Big|_{z=z_0} = \lim_{h \to 0} \frac{e^{z_0+h} - e^{z_0}}{h} = \lim_{h \to 0} e^{z_0} \frac{e^h - 1}{h}.$$

Let $h = \sigma + i\tau$, with $\sigma, \tau \to 0$. Then

$$\frac{e^h - 1}{h} = \frac{e^\sigma(\cos\tau + i \sin\tau) - 1}{\sigma + i\tau}.$$

Now expand each term for small σ, τ:

$$e^\sigma = 1 + \sigma + O(\sigma^2), \qquad \cos\tau = 1 - \frac{\tau^2}{2} + O(\tau^4), \qquad \sin\tau = \tau + O(\tau^3).$$

Hence,

$$e^\sigma(\cos\tau + i\sin\tau) = (1+\sigma)\Big(1 - \tfrac{\tau^2}{2} + i\tau\Big) + O(\sigma^2).$$

Expanding:

$$e^\sigma(\cos\tau + i\sin\tau) = 1 + \sigma + i\tau - \tfrac{\tau^2}{2} + O(\sigma\tau, \tau^2, \sigma^2).$$

Therefore,

$$e^\sigma(\cos\tau + i\sin\tau) - 1 = \sigma + i\tau + O(\sigma^2, \tau^2, \sigma\tau).$$

So

$$\frac{e^h - 1}{h} = \frac{\sigma + i\tau + O(\sigma^2, \tau^2, \sigma\tau)}{\sigma + i\tau} \quad \longrightarrow \quad 1.$$

Finally,

$$\frac{d}{dz} e^z \Big|_{z=z_0} = e^{z_0} \cdot 1 = e^{z_0}.$$

Definition: **Cauchy-Riemann Equations**:
If f(z) is differentiable then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

exists for any sequence $h \in \mathbb{C}$, $h \to 0$.
Suppose $f = u + iv$ is analytic in D then

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Consequence:

Assume f = u + iv is analytic in D. Assume u,v have continuous derivatives up to order 2. Then if $\nabla^2 u, \nabla^2 v = 0$, and following the C-R equation, then u and v are said to be **Harmonic Conjugates**.

Here is also a converge of the C-R equations

Theorem: Let f = u + iv and assume that u,v,$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are all continuous in a disc centered at z_0. If u and v satisfy the C-R equation at z_0, then f is differentiable at z_0, and

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}}$$

the proof is on page 10 of the note.

5 Power Series

Theorem: Consider the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $0 < R \leq \infty$, and converging on $|z - z_0| < R$
Then f(z) is analytic in the disc $\{|z - z_0| < R\}$ and

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^n$$

see the proof on Lecture 7 note Page 2 - 5. On Page 3, we used triangle inequality, and let z strictly less than r.

Theorem: if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence $0 < R \leq \infty$, then in $\{|z - z_0| < R\}$, f(z) is infinitely differentiable and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)...(n-(k-1)) a_n (z - z_0)^{n-k}$$

Therefore, $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ by setting z = z_0

Here are few ways to find radius fo convergence to put in mind

- if $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exsits, then

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{R}$$

- If $\lim_{n\to\infty} (|a_n|)^{1/n}$ exists then,

$$\frac{1}{R} = \lim_{n\to\infty} (|a_n|)^{1/n}$$

Note: that a conclusion

$$n r^{n-1} \leq s^n$$

is obtained for r < s < R and by using the ratio test of $\lim_{n\to\infty} n(r/s)^n = 0$ we have that at large $n \geq N$ we have $n(r/s)^n \leq 1$, which proves that $nr^n \leq s^n$. Therefore, we may write that

$$\sum_{n=1}^{\infty} n|a_n|r^{n-1} \leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=N}^{\infty} |a_n|s^n, \quad (5)$$

$$\leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=1}^{\infty} |a_n|s^n, \quad (6)$$

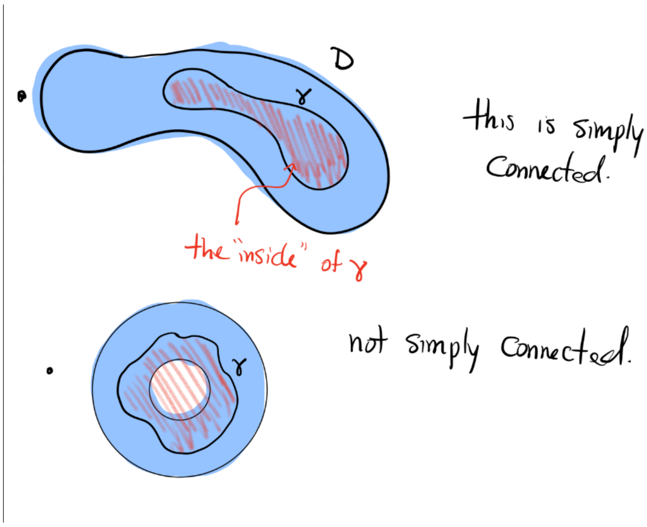
$$\text{and} \quad \sum_{n=1}^{\infty} |a_n|s^n \text{ converges since } s < R. \quad (7)$$

which proves that $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ converges.

6 Cauchy Theorem

Let starts with some definition first.

Definition: A domain D is **simply-connected** if, whenever γ is a simple closed Curve in D, the inside of γ is also a subset of D.



Here are some examples of how D is simply-connected, or in simpler language, the domain doesn't have any holes. With this definition we start the Cauchy's Theorem

Theorem: Cauchy's Thm
Suppose f is analytic on a domain D. Let γ be a piecewise C^1, simple closed curve in D st. the inside of $\gamma = \Omega \subseteq D$. Then

$$\int_\gamma f(z)dz = 0$$

The proof is just the combination of Green's Theorem and the Cauchy-Riemann Equation. However, the theorem still holds if γ is not simple.(not intersecting itself)

Theorem: if D is simply connected domain and f is analytic on D, then there is an analytic function F on D st.

$$F' = f$$

proof can be seen on page 10 of Lecture 7. Essentially, taking a path γ from $z_0 \to z_1$, and taking another path γ_1 going in reverse, the curve defined by this loop can be defined by

$$0 = \int_\Gamma f dz = \int_\gamma f dz - \int_{\gamma_1} f dz$$

Then we can prove that F is differentiable as shown on page 12.

Theorem: Cauchy's Integral Formula Suppose f is analytic on a domain D, γ is piecewise C^1, positively oriented, simple closed curve st inside $\gamma = \Omega \subseteq D$. Then, $\forall z \in \Omega$,

$$\boxed{f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \quad \forall z \in \Omega}$$

This integral has many applications, for instance, we may have

Example 7: Compute $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$
Idea: write this as an integral for an analytic function over the circle $z = e^{i\theta}, 0 \le \theta \le 2\pi$. If $|z| = 1$m then

$$\sin\theta = (\frac{z - z^{-1}}{2i})$$

so the integral becomes

$$\gamma(\theta) = e^{i\theta}$$

$$\int_\gamma \frac{1}{2 + \frac{z-z^{-1}}{2i}} \frac{dz}{iz} = \int_\gamma \frac{2\,dz}{4iz + (z^2-1)}.$$

$$z^2 + 4iz - 1 = \left(z - \frac{-4i + \sqrt{(-4i)^2 - 4}}{2}\right)\left(z - \frac{-4i - \sqrt{(-4i)^2 - 4}}{2}\right)$$

$$= (z - i(\sqrt{3} - 2))\,(z + i(\sqrt{3}+2)).$$

$$\text{Since } |\sqrt{3} - 2| < 1, \quad \sqrt{3} + 2 > 1, \quad \text{we can apply the Cauchy integral formula.}$$

$$\int_\gamma \frac{2\,dz}{(z - i(\sqrt{3}-2))(z + i(\sqrt{3}+2))}$$

By the Cauchy Integral Formula:

$$= 2\pi i \cdot \frac{2}{((\sqrt{3}-2)i + (\sqrt{3}+2)i)}.$$

$$= 2\pi i \cdot \frac{2}{2\sqrt{3}\,i} = \frac{4\pi i}{2\sqrt{3}\,i} = \frac{2\pi}{\sqrt{3}}.$$

There is a very important theorem for Cauchy-riemann theorem

Theorem: if f(z) is analytic in a domain D, $z_0 \in D$, and $\{|z - z_0| < R\} \subseteq D$, then f has a convergent power series expansion in this.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where a_k is determined by an integral formula

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

and $\gamma = \{|z - z_0| = R\}$ is positively oriented.

in other word, if f is analytic on D, then so. is f', which means f is infinitely differentiable. Proof of the theorem above can be found in Lecture 8, Page 3 in the google drive note. The corollary is that, in the setting of Thm

$$\frac{f^k(z_0)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

in particular, if f is analytic in a domain D. All the derivative vanish at some point. $f^k(z_0) = 0, \forall k$ at some $z_0 \in D$, then f $= 0$ on D. So called the **Unique Analytic Continuation**. In comparison with real function theory on the same theorem. Consider

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Here is also an exercise, f is infinitely differentiable at x = 0 and $f^{(k)}(0) = 0$. We notice that $f \neq$ Taylor series of f at 0 on any ball containing 0.

6.1 The Order of a Zero

Definition: Suppose f is analytic in a disc D, f is not identically zero, and $f(z_0) = 0$ for some $z_0 \in D$, then

$$f = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

let $m \geq 1$ be the smallest n st. $a_n \neq 0$. That is

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + ...$$

we say f has a **zero of order** m at z_0. Then the function

$$g(z) = \frac{f(z)}{(z - z_0)^m}$$

is analytic in D.

There is also a partial converse theorem to Cauchy's Thm.

Theorem: If f is continuous in a domain D and $\int_\gamma f(z)dz = 0$, for every triangle γ st $\gamma \subseteq D$ and inside(γ) $\subseteq D$, then f is analytic in D.

The application of Cauchy's Theorem requires some other theorems to back it up

Theorem: **Liouville's Theorem**:
If F is entire(analytic over the entire complex plane), and $|F(z)| \leq M$, then F is constant.

The proof is in lecture 9 Page 7.

6.2 Analytic Logarithms

Theorem: Let D is a simply connected domain. Suppose f is analytic in D and $f \neq 0$ anywhere in D. Then $\frac{f'(z)}{f(z)}$ is analytic and hence so is

$$h(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

where the integral is over any path from z_0 to z. Meaning h(z) is simply path independent. The proof is that

$$h'(z) = \frac{f'(z)}{f(z)}$$

and thus

$$[e^{-h(z)} f(z)]' = -e^{-h(z)} \frac{f'(z)}{f(z)} f(z) + e^{-h(z)} f'(z) = 0 \to e^{-h(z)} f(z) = c = f(z_0)$$

Thus $g(z) = h(z) - Log(f(z_0))$ meaning that given a z and a z_0 the value of h(z) = $g(z) + Log(f(z_0))$

6.3 Isolated Singularities

Definition: An Analytic function has an isolated singularity at z_0 if it is analytic in a punctured disc $\{0 < |z - z_0| < r\}$ for some r > 0. We may also say

- z_0 is a **Removable Singularity** if $|f(z)|$ is bounded near $z \to z_0$.
- z_0 is a **pole** if $|f(z)| \to \infty$ as $z \to z_0$. if $f(z) = \frac{H(z)}{(z - z_0)^m}$, $H(z)$ is analytic on $\{|z - z_0| < r\}$ and $H(z) \neq 0$, then we say f(z) has a pole of order m at z_0.
- z_0 is an **essential singularity** if neither (i) nor (ii) hold.

6.4 Residue

Imagine a function f(z) is analytic everywhere except at one point z_0. Drawing a circle aroundz_0 say $|z - z_0| = s$, the residue of f is defined as

$$Res(f; z_0) = \frac{1}{2\pi i} \oint_{|z - z_0| = s} f(z)dz$$

f(z) can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Note that the residue can be understood as the non-analytic portion of the function f(z). A great intuitive understanding is that by the Cauchy's theorem, all the analytic parts of the function f(z) vanish by drawing a circle, therefore all that is left is the non analytic part.

recall in Homework 2, we proved that only when n = -1 $\int (z - z_0)^n dz = 2\pi i$. Therefore, only the a_{-1} term survives meaning

$$Res(f; z_0) = a_{-1}$$

in other word, the residue is the coefficient of $\frac{1}{z - z_0}$ in the expansion of the analytic function. One way to find the residue is that, for a function that has a simple pole, it can be expands as

$$f(z) = \frac{b_{-1}}{z - z_0} + b_0 + b_1(z - z_0)...$$

multiplying the singularity term $z - z_0$

$$(z - z_0)f(z) = b_{-1} + b_0(z - z_0) + b_1(z - z_0)^2$$

taking the limit to $z \to z_0$, we have only the b_{-1} term left. Here are a few examples that are important for finding the residue

Example 8:

$$\frac{e^z - 1}{z^2} \quad at \quad z_0 = 0$$

Solution:

$$e^z = 1 + z + \frac{z^2}{2} + ...$$

$$\frac{e^z - 1}{z^2} = \frac{z + z^2/2 + ...}{z^2} = 1/z + 1/2 + ...$$

Therefore Res = 1.

Another example is

Example 9: Find the residue of

$$\frac{(z^2+3z-1)}{z+2}$$

and its pole
Solution:
pole is at z = -2 / we may rewrite the numerator in terms of z + 2 = w

$$\begin{aligned}(z^2+3z-1) &= ((w-2)^2+3(w-2)-1)\\ &= w^2-4w+4+3w-6-1\\ &= w^2-w-3\end{aligned}$$

Therefore, the coefficient before m = -1 term is -3 => Res = -3

There is also an important theorem with residue

Theorem: The Residue Theorem
suppose f is analytic on a simply connected domain D, except for a finite number of isolated singularities at $z_1, ..., z_n \in D$. Let γ be a piecewise C^1, positively oriented, simple closed curve that does not pass through any of the point $z_1, ...z_n$ Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_k} Res(f; z_k)$$

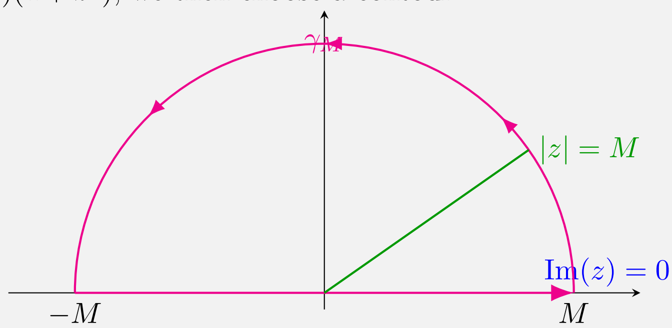
where z_k is inside γ

This is more like an application of Cauchy Integral formula, and we may solve some interesting problems with this

Example 10: Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}$$

Solution:
let $p(z) = z^2, q(z) = (1+z^2)(4+z^2)$, we then choose a contour



suppose M is a large number we may find the solution to the integral.

$$\int_{\gamma_M} \frac{p(z)}{Q(z)} = \int_{-M}^{M} \frac{x^2}{(1+x^2)(4+x^2)} + \int_0^\pi \frac{p(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta} d\theta$$

note that the second term scales with M^3/M^4, therefore for large M it goes to 0. The remaining term can be now solved using Residue formula. Q(z) has zeroes at $z = \pm i, \pm 2i$. Yet only $i, 2i$ are inside γ_M for M large. Therefore

- $z = i$

$$\lim_{z\to i} \frac{z^2}{(z+i)(z-i)(z^2+4)} = \lim_{z\to i} \frac{1}{(z-i)}[\frac{z^2}{(z+i)(z^2+4)}] = \frac{-1}{6i}$$

- $z = 2i$

$$\lim_{z\to i} \frac{z^2}{(z+2i)(z-2i)(z^2+1)} = \lim_{z\to i} \frac{1}{(z-2i)}[\frac{z^2}{(z+2i)(z^2+1)}] = \frac{1}{3i}$$

summing them up, we ahve

$$\int_{\gamma_M} \frac{p(z)}{Q(z)} = 2\pi i(\frac{-1}{6i} + \frac{1}{3i}) = \frac{2\pi}{6}$$

from which we have few proposition. P,Q polynomials that are real valued on Im(z) = 0, and st. $degQ \geq degP + 2$, then we may use the solution above.

Here we have some examples for integrals involving trigonometric functions.

Example 11: Compute $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+\alpha^2} dx$ where $\alpha > 0$.

step1. we replace the integrand with $\frac{e^{iz}}{z^2+\alpha^2}$, and we use the same contour as before. Therefore we have, for the magnitude.

$$|e^{iz}| = e^{i(z-\overline{z})/2} = e^{-M\sin\theta}$$

Therefore

$$|\int \frac{e^{iz}}{z^2+\alpha^2} dz| \leq M \int_0^{\pi} \frac{e^{-M\sin\theta}}{M^2-\alpha^2} d\theta \to 0$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+\alpha^2} dz = \lim_{M\to\infty} \int_{\gamma_M} \frac{e^{iz}}{z^2+\alpha^2} dz$$

since $z^2+\alpha^2$ has zeros at $z = \pm i\alpha$, we have the residue as

$$Res(f; i\alpha) = \frac{e^{-\alpha}}{2i\alpha}$$

Therefore

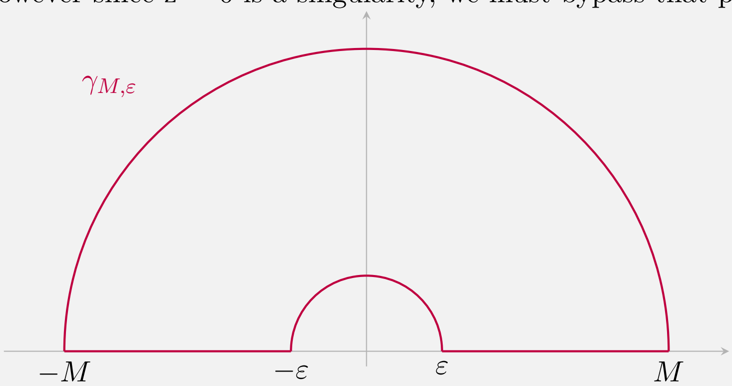
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2} = 2\pi i \frac{e^{-\alpha}}{2i\alpha} = \frac{\pi}{\alpha} e^{-\alpha} = Re(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2}) = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+\alpha^2}$$

Here is also an important example

Example 12: Compute $\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx$
we first replce the function with

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{1}{2} \int_0^{\infty} \frac{(1-\cos(2x))}{x^2} dx$$
$$= \frac{1}{4} \int_{-\infty}^{\infty} \frac{Re(1-e^{2ix})}{x^2} dx$$

we now need a contour, however since z = 0 is a singularity, we must bypass that place



By Cauchy Integral formula, we have that

$$0 = \int_{\gamma_{M,e}} f(z)dz = \int_{\{z=Me^{i\theta}\}} f(z)dz + \int_{\{z=\epsilon e^{i\theta}\}} f(z)dz + \int_{-M}^{-\epsilon} f(z)dz + \int_{M}^{\epsilon} f(z)dz$$

The first integral is

6.5 Laurent Series

Suppose f is analytic in two overlap punctured disc or the annulus $0 < r < |z - z_0| < R$. Does f admit some sort of power series?

Theorem: If f is analytic on $0 \leq r < |z - z_0| < R$, then we can write

$$f(z) = f_1(z) + f_2(z)$$

where

1. $f_1(z)$ is analytic on $\{|z - z_0| < R\}$
2. $f_2(z)$ is analytic on $\{|z - z_0| > r\}$ including at ∞

in particular

$$f_1(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
$$f_2(z) = \sum_{k=1}^{\infty} b_k (z - z_0)^{-k}$$

combining them together we have

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$$

where $a_k = b_{-k}$ for $k < 0$, and this function is valid on $r < |z - z_0| < R$

Let say the question asks you to define laurent series for $z < 1$ this range sets the radius of convergence for the series you expand, all you really need to do is to perform a taylor expansion of the function.

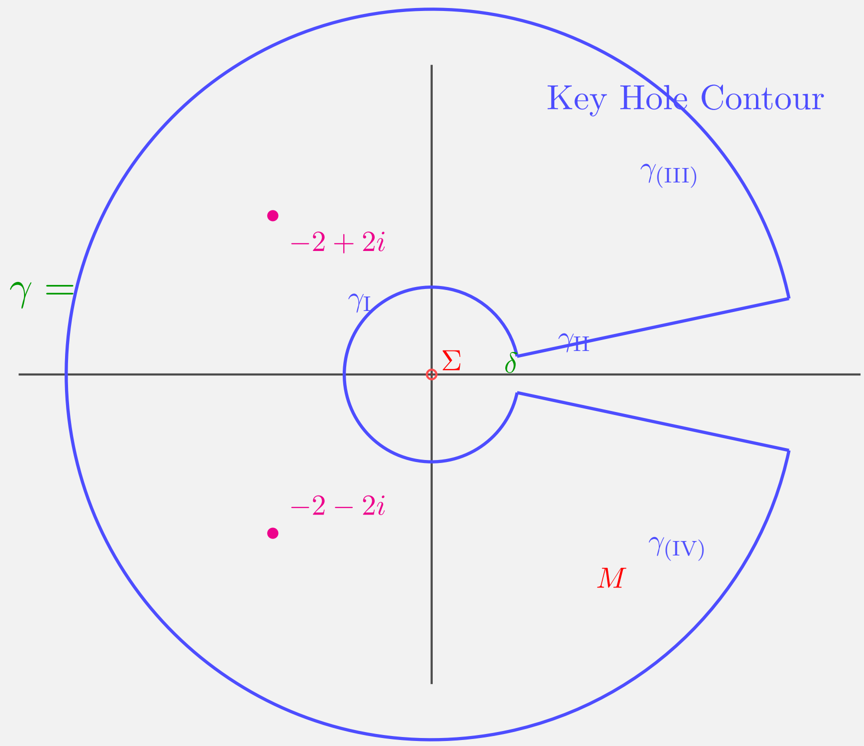
6.6 More Contour Integral

There are some questions that need complex contour shape. For instance, when we are asked to find such integral

Example 13: find the value of

$$\int_0^{\infty} \frac{x^{1/3}}{x^2 + 4x + 8} dx$$

since the integral is not an even function, the contour we used previously no longer works. Therefore, we need



this key hole contour. By the normal derivation, w efind that only γ_{II}, γ_{IV} stays, so we may find that

$$\int_{\gamma_{IV}} f(z)dz = \int_M^\epsilon \frac{r^{1/3}e^{i(2\pi-\delta)/3}e^{i(2\pi-\delta)}dr}{((re^{i(2\pi-\delta)})^2 + 4(re^{i(2\pi-\delta)}) + 8)} \tag{8}$$

and

$$\int_{\gamma_{II}} f(z)dz = \int_\epsilon^M \frac{r^{1/3}e^{i\delta/3}e^{i\delta}dr}{((re^{i\delta})^2 + 4(re^{i\delta}) + 8)} \tag{9}$$

Therefore, we may find that as $M \to \infty$

$$\int_0^\infty \frac{r^{1/3}dr}{r^2+4r+8}[1-e^{i2\pi/3}] = \int_\gamma f(z)dz = 2\pi i \sum Res$$

6.7 Zeros of an Analytic function

Theorem: If f(z) is analytic near z_0, f not identically zero, and $f(z_0) = 0$, then we can write

$$f(z) = (z-z_0)^m g(z) \quad m \geq 1$$

where $g(z)$ is analytic near z_0 and $g(z_0) \neq 0$ m is the order of the zero of f at z_0.

Theorem: Suppose h is analytic in a domain D except for a finite number of poles. Let γ be a piecewise continuously differentiable, positively oriented, simple closed curve in D, which does not pass through any pole or zero of h, and s.t. inside(γ) $\subseteq D$, then

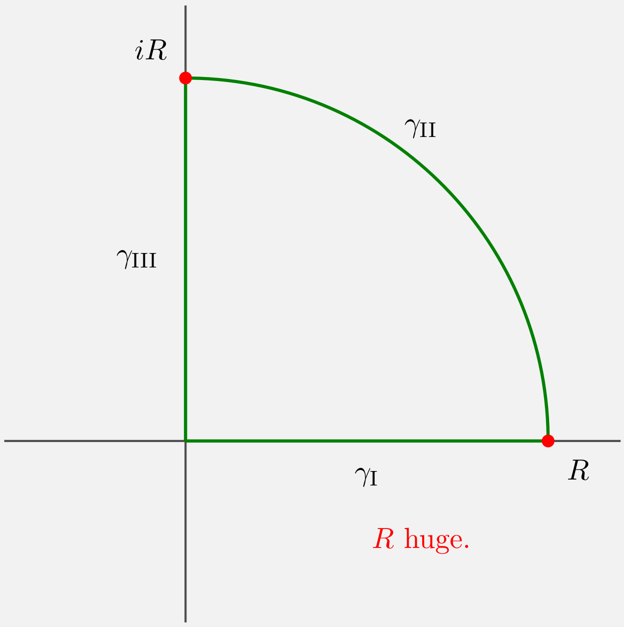
$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz = \#\text{ of zeros in side } \gamma - \#\text{ of poles inside } \gamma$$

For more details check lecture 13 for a quick review.

Theorem: The Argument Principle
Suppose h is analytic in a domain D except for a finite number of poles. Let γ be a piecewise continuously differentiable, positively oriented, simple closed curve in D, which does not pass through any pole or zero of h, and s.t. inside(γ) $\subseteq D$, then

$$\frac{1}{2\pi}\{\text{change in arg } h(z) \text{ as z travels } \gamma\} = \{\text{\# of zeros of h inside } \gamma\} - \{\text{\# of poles of h inside } \gamma\}$$

using the argument principle we may solve some questions.



Example 14: Consider the contour we may find the number of zeros inside here by going along the border.

- X-axis, since the function f $= x^3 - x^2 + 4$, f' $= 3x^2 - 4x$, f(4/3) $= 4 - \frac{32}{27} > 2$, so there is no argument change.
- y axis: f $= -iy^3 + 2y^2 + 4$, as y goes large, we find that $-iy^3$ dominates so it goes to $-\pi/2$. The change in arguement is then $\pi/2$
- Along the curve: we define $z = Re^{it}$, we may find that R^3e^{3it} dominates and thus, the angles change by $3t$ as t goes to $\pi/2$. Therefore, the argument change is $3\pi/2$.

in Sum the total change in argument is 2π and the number of zeros is $\frac{1}{2\pi} * (2\pi) = 1$

Theorem: Rouche's Theorem
suppose f,g are analytic on D, γ a curve in D(piecewise continuously differentiable, simple closed) If

$$|f(z) + g(z)| < |f(z)| \quad \forall z \in \gamma$$

Then f and g have the same number of zeros inside γ (proof can be found on Lecture 14)

Theorem: The Fundamental Theorem of Algebra
Suppose $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ is a complex polynomial. Then P(z) has n zeros counting multiplicity

6.8 Maximum Modulus

Recall that if f is analytic on a domain D, then either

1. f is constant
2. f(D)$\subseteq \mathcal{C}$ is open

There is also an important corollary for this that if f is a non-constant analytic function on a domain D and $f(z)-f(z_0)$ has a zero of order m at z_0, then near z_0 the map f is $m \to 1$. In particular, if $f'(z_0) = 0$, then the zero of order of $f(z) - f(z_0)$ is at least $m \geq 2$.
Similarly, the **Maximum modulus principle** reads that: if f is a non-constant analytic function on a domain D, then |f| has no local max on D.

Theorem: **Schwarz Lemma**
suppose f is analytic in a disc |z| < 1, f(0) = 0 and |f(z)| ≤ 1, $\forall |z| < 1$, then

$$|f(z)| \leq |z| \quad \forall |z| < 1$$

and $|f(z)| = z$ for some $z \neq 0$ if and only if $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$

For the mean value properties of an analytic function.

Theorem: Suppose $f = u + iv$ is analytic on $\{|z - z_0| \leq r\}$ then, for any $s \leq r$

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + se^{i\theta}) d\theta = \{\text{Average value of u on the circle}\}$$
$$v(z_0) = \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + se^{i\theta}) d\theta$$

6.9 Linear Fractional Transformations

Definition: A linear Fractional Transformation is a rational function of the form

$$T(z) = \frac{az + b}{cz + d} \quad ad - bc \neq 0$$

such that T is one to one transformation, T has an inverse T^{-1} which is also a linear Fractional Transformation, if T_1, T_2 are linear Fractional transformation then $T_1 \cdot T_2(z) = T_1(T_2(z))$ is a linear fractional transformation. A matrix representation of this is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_T$$

and A_T is invertible also $A_{T_1 \cdot T_2} = A_{T_1} A_{T_2}$. It can also be thought of in terms of how it moves points on S^2

- **Fixed Point**: A fixed point of T is a solution of $T(z) = z$, and a fractional linear transformation either has ≤ 2 fixed points or is the identity matrix.
- **Uniqueness**: Lets say three distinct z points and three distinct w solution, there is a unique T for each z mapping to each w.

7 Residues: Laurent series

The residue at z_0 is defined as

Definition: suppose f is analytic on $0 < |z - z_0| < r$ if $0 < s < r$ define

$$Res(f, z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} f(\zeta) d\zeta$$

8 Conformal Mapping