Personal Notes on

E&M (ECE259)

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Abstract

This note is documented based on the class instructed in ECE259: Electromagnetism. Also the note reference a lot of concepts from Griffith's Introduction to Electromagnetism, as well as Steck's Classical and Modern Optics

Contents

1 Electrostatics

1.1 Electric Field in Matter

Definition: Perfect Dielectrics
There are no free charges, All charges(atoms) are bounded.

Given all charges are bounded, the atoms formed a dipoles and induced polarization under an external field, the induced potential given a polarization vector(or dipole moment per unit volume)$\overline{p_k}$ is:

$$V_k \simeq \frac{\vec{p}_k \cdot (\vec{R} - \vec{R}'_k)}{4\pi\varepsilon_0 |\vec{R} - \vec{R}'_k|^3} \quad \text{for } R \gg d_k$$

let $\overline{P}$ denote the total dipole moment of the solid, the voltage from the entire object becomes

$$V = \int_{V'} \frac{\vec{p}_k \cdot (\vec{R} - \vec{R}'_k)}{4\pi\varepsilon_0 |\vec{R} - \vec{R}'_k|^3} dV'$$

and through divergence theorem we have the total voltage as

$$V = \int_{S'} \frac{\vec{P} \cdot \vec{a}'_n}{4\pi\varepsilon_0 |\vec{R} - \vec{R}'|} dS' + \int_{v'} \frac{-\nabla' \cdot \vec{P}}{4\pi\varepsilon_0 |\vec{R} - \vec{R}'|} dv'$$

$$\rho_{P,S} = \vec{P} \cdot \vec{a}'_n \quad \text{Surface Charge Denstiy}$$

$$\rho_{P,V} = -\nabla' \cdot \vec{P} \quad \text{Volume Charge Density}$$

The first term looks like the voltage due to the surface charge density and the second term looks like the volume charge density. This voltage is reference to infinity, induced by the dipoles of an object. Essentially, polarization produces surface bound charge and volumetric charge.

1.2 Generalized Gauss Law

There are two charges in consideration: Bound charge and Free charge. The first one stands for the charges induced by polarization, and the second one is everything else, either electrons or ions embedded in the dielectric material that is not a result of polarization.

$$\rho = \rho_f + \rho_{P,V}$$

The gauss law reads

$$\nabla \cdot (\epsilon_0 E + P) = \rho_f = \nabla \cdot D = \rho_f$$

or integral form

$$\oint D \cdot da = Q_{fenc}$$

where D is known as the electric displacement, and Q_{fenc} stands for the total free charge enclosed in the volume.

A related constant to this topic si the **electric Susceptibility** which is a measurement of the tendency of a material to be polarized.

$$\vec{P} = \underbrace{\varepsilon_0 \chi_e}_{\text{Electric Susceptibility}} \vec{E}$$

such that the model describes polarization as

- Linear to External Field
- Isotropic

$$D = \epsilon_0 (1 + \chi_e) E = \epsilon E$$

where ϵ is the permittivity of the material, and $\epsilon_r = 1 + \chi_e = \dfrac{\epsilon}{\epsilon_0}$ is the relative permittivity or dielectric constant of the material.

Boundary Conditions

This is a supplementary section. E, D, P can be discontinuous at an interface.

$$\begin{cases} E_{1,t} = E_{2,t} \\ D^{\perp}_{above} - D^{\perp}_{below} = \sigma_s \end{cases}$$

where σ_s stands for the density of free charge at interface.

1.3 Capacitance

A simple definition is that

> **Definition: Dielectric Strength**:
> Maximum |E| the material can withstand without breaking down.

Here are a few equations to know about capacitor

- Charge:

  $$Q = CV$$

  where V is the voltage between two objects, and Q is the common charge on each object.

- Capacitance: The capacitance of a plate capacitor is

  $$C = \epsilon \frac{A}{d}$$

  A is the area of the plate and d is the distance between the plates.

- Stored Energy: A preliminary knowledge, the energy of a N multi-particle system is

  $$W_e = \frac{1}{2} \sum_{k=1}^{N} q_k \cdot V_k$$

  where V_k stands for the potential produced by all charges except q_k at R_k, continuously

  $$W_e = \int_{v'} \frac{1}{2} \rho_v(r) V(r) dv'$$

  Alternative expression is

  $$W_e = \frac{1}{2} \int (\nabla \cdot D) V dv' = \frac{1}{2} \int \epsilon |E|^2 dv'$$

  which we can use this equation to find the energy of a plate capacitor

  $$W_e = \frac{1}{2} \int \epsilon \frac{V^2}{d^2} dv = \frac{1}{2} \frac{\epsilon A}{d} V^2 = \frac{1}{2} QV = \frac{1}{2} CV^2$$

1.4 Poisson Equation

Through transforming the equation of gauss's law in polarization format we have the **Poisson Equation**

$$\nabla \cdot (\epsilon \nabla V) = -\rho_v$$

if $\rho_v = 0$ is the Laplace Equation. The use of poisson is as follows

> **Example 1:** Finding V by solving a boundary value problem, with V_0 at the boundary and ρ_v inside the material. Find V inside the volume.

Answer:

$$\begin{cases} \nabla \cdot (\epsilon \nabla V) = -\rho_v \\ V_S = V_0 \end{cases}$$

you can find V by doing an integral

Or we can look into specific case of a capacitor

Example 2: Where

$$\begin{cases} \nabla \cdot (\epsilon \nabla V) = 0 \\ V(y = d) = V_0 \\ V(y = 0) = 0 \end{cases}$$

Answer: By solving the Poisson Equation

$$\epsilon \nabla^2 V = 0 \to V(y) = c_1 y + c_2$$

then

$$\begin{cases} V(d) = c_1 d + c_2 = V_0 \\ V(0) = c_2 = 0 \end{cases}$$

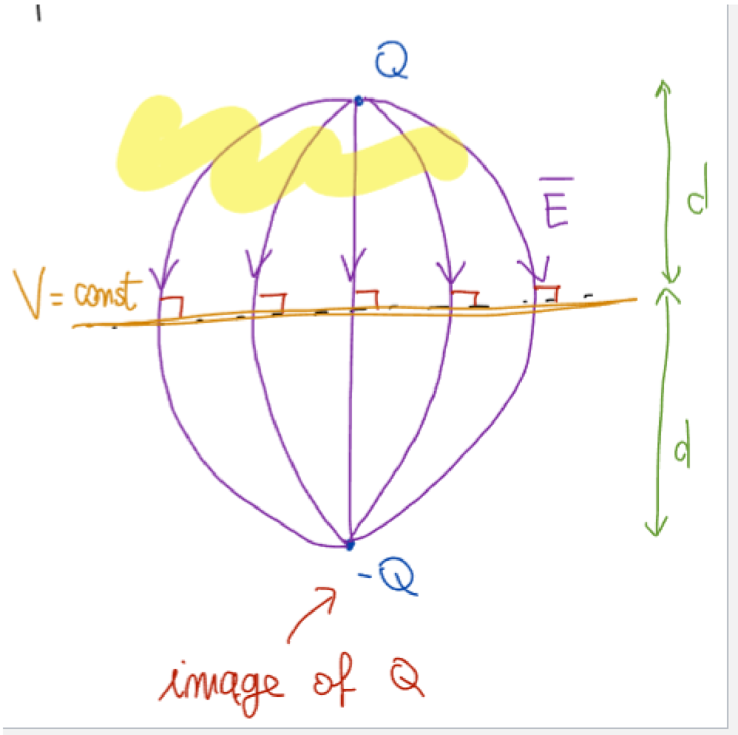
so

$$V(y) = \frac{V_0}{d} y$$

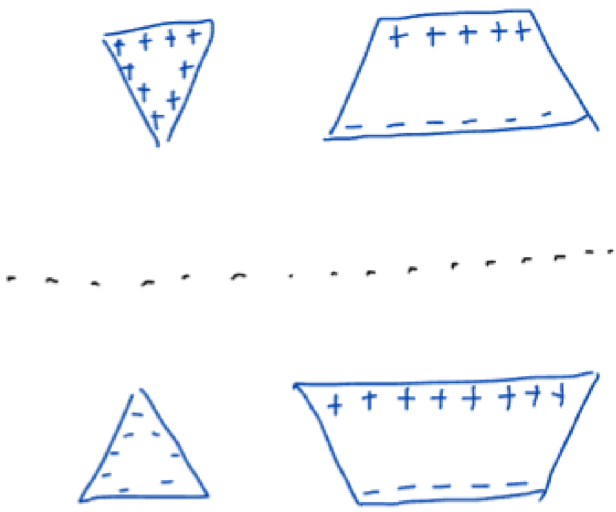
Theorem: Uniqueness Theorem
Given ρ_v, V_0 the solution fo the poisson equation is unique.

1.5 Image Charge/Theory

How to find the effect of a electric charge if the charge is right above a perfect conductor that is hard to model its' charge distribution. We can do that by thinking there's a charge on the other side of the interface. For irregular



shape it is like this



1.6 Current & Resistance

Current is the coherent motion of all free electrons, where

$$u_e = -\mu_e \overline{E}$$

where u_e is the drift velocity of electrons and μ_e is the electrons mobility $[\frac{m^2}{V \cdot s}]$ The current is the current density through a surface S

$$I = \frac{dq}{dt} = \int_s \rho \overline{u} \cdot d\overline{S} = \int_s \overline{J} \cdot d\overline{S}$$

The origin of the microscopic Ohm's law is through here.

$$\overline{J} = (-\mu_e \rho_e)\overline{E} = \sigma \overline{E}$$

where σ stands for the conductivity $[S/m]$, and the macroscopic resistance is defined as

$$R = \frac{V}{I}$$

1.7 Continuity Equation/Joule Law

by charge conservation through a surface

$$q_{in} + q_{out} = c$$

by Divergence Theorem we get the Continuity Equation

$$\frac{\partial \rho_v}{\partial t} + \nabla \cdot \overline{J} = 0$$

or integral form

$$\frac{\partial}{\partial t} \int \rho_v dv + \int \overline{J} \cdot \overline{dS} = 0$$

We can also get **Relaxation Equation** through a combination with Gauss Law

$$\frac{\partial \rho}{\partial t} + (\frac{\sigma}{\epsilon})\rho(t) = 0 \rightarrow \rho(t) = \rho(t=0)e^{-t/\tau}$$

where $\tau = \epsilon/\sigma$ is the relaxation time, describing the distribution of charge density of an uniformly distributed object as a function of time.

Joules' Law

This defines the work done by E to move dq by distance dl

$$dW = dq\overline{E} \cdot \overline{dl} = \rho dv \overline{E} \cdot \overline{dl}$$

which we get

$$\frac{dW}{dt} = P = \int_v \overline{J} \cdot \overline{E} dv = IV$$

Also consider the situation of an interface under steady-state conditions

$$\begin{cases} \nabla \cdot J = 0 \\ \nabla \times (J/\sigma) = 0 \end{cases}$$

at the interface, the current density at the top and bottom of the surface

$$J_{top} = J_{bottom}$$

and

$$\frac{J_{top}}{J_{bottom}} = \frac{\sigma_{top}}{\sigma_{bottom}}$$

2 Potential

2.1 Laplace & Poisson Equation

The formula that governs potential configurations are

- **Laplace Equation**:
$$\nabla \cdot (\epsilon \nabla V) = 0$$

- **Poisson Equation**:
$$\nabla \cdot (\epsilon \nabla V) = \rho_v$$

also the **Uniqueness Theorem** says that given the set boundary condition and charge density, the solution to Poisson Equation is unique.

2.2 Cartesian: Separation of Variables

The most typical way to solve Poisson Equation in Cartesian coordinate is through the **Separation of Variables** which is essencially

1. Set $V(x, y) = X(x)Y(y)$
2. Sub into $\nabla^2 V = 0$
3. Obtain the expression
$$\frac{d^2X}{dx^2} = k^2 X, \quad \frac{d^2Y}{dy^2} = -k^2 Y$$
such that
$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$
Ex: Lets say the final solution for a condition yields
$$V(x, y) = Ce^{-kx} \sin ky = Ce^{-kx} \sin n\pi y/a$$
The solution for $V(x, y)$ is a linear combination with all possible n = 1,2,3...
$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$
to find the coefficient for each component, we can use Fourier Trick:
$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy$$

2.3 Spherical: Legendre Polynomials

The laplace Equation in spherical coordinates reads

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Yet, if we assume there is an azimuthal symmetry, then we can ignore the ϕ term

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) = 0$$

such that

$$V(r, \theta) = R(r)\Theta(\theta)$$

by the separation of variables we have

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = l(l+1), \quad \frac{1}{\Theta \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = -l(l+1)$$

and the difficult part about the solution is the angular equation

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\Theta$$

where the solution are **Legendre Polynomials** in the variable $\cos\theta$

$$\Theta(\theta) = P_l(\cos\theta)$$

and the polynomails is defined by the **Rodrigues Formula**

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

the first few solutions of them follow the table:

$$\boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \\ P_4(x) &= (35x^4-30x^2+3)/8 \\ P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned}}$$

therefore, the solution of the potential follows that

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

Example 3: The potential $V_0(\theta)$ is specified on the surface of a hollow sphere of radius R. Find the potential inside the sphere.
Solution:
$B_l = 0$ other wise the solution will blow up at the origin. At r = R the solution must match the specified function

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta).$$

By fourier Trick, which is slightly different for legendre polynomials.

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases}$$

However, it is hard to solve for the coefficient analytically so it is easier to eyeball it. For instance

$$V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)] = \sum A_l R^l P_l(\cos\theta)$$

The legendre polynomial is also used for multi pole expansion, through some binomial expansion and substitution, we can find that

$$\frac{1}{\Delta r} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(\cos\alpha)$$

where α is the angle between the charges and r is the distance between origin to the destination and r' is the distance between the origin to the charge. Therefore

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\mathbf{r}')\,d\tau',$$

This essentially describes the potential of an object induced at a distance assuming a certain charge density of the object. Also some additional information

- **Dipole Moment:** For a continuous object with some charge distribution, assume that the total charge is zero, the dominent term in the potential will be dipole potential

  $$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int r'\rho(\mathbf{r}')\,d\tau'$$

  where

  $$p = \int r'\rho(\mathbf{r}')\,d\tau' = \sum_{i=1}^{n} q_i r_i'$$

  is the dipole moment of the charge distribution or point charges(second formula)

3 Magnetism

There are some equations to draw analogy with electric cases

- Force:

$$F_B = qu \times B$$

- Continuity Equation

$$\nabla \cdot B = 0$$

$$\nabla \times B = \mu_0 J$$

which is the Ampere's Law, where $\mu_0 = 4\pi \times 10^{-7} H/m$ or in integral form

$$\oint B \cdot dS = 0$$

$$\oint B \cdot dl = \mu_0 I$$

where I is the current enclosed by the path and the integral is the close-path integral of B field.

- Magnetic Vector Potential:

$$\nabla \cdot B = 0 \rightarrow B = \nabla \times A$$

where A is the magnetic vector potential such that $\nabla \cdot A = 0$

Using poisson equation on the vector potential we can have

$$\nabla^2 A = -\mu_0 J$$

and the superposition integral of the magnetic vector potential created by an object is

$$A = \frac{\mu_0}{4\pi} \int_v \frac{J(R')dv'}{|R - R'|}$$

or created by a line current

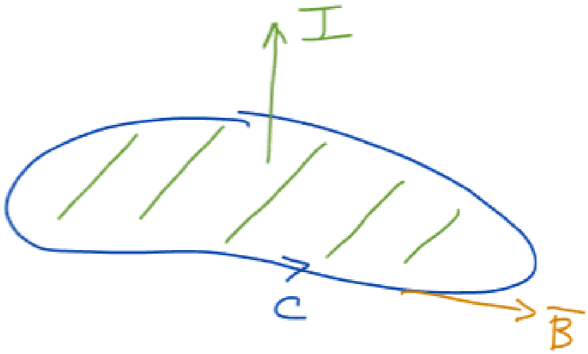
$$A = \frac{\mu_0 I}{4\pi} \int \frac{dl}{|R - R'|}$$

This gives the well-known **Biot-Savart Law**

$$B = \nabla \times A = \frac{\mu_0 I}{4\pi} \int_{c'} \frac{dl' \times (R - R')}{|R - R'|^3}$$

The last thing to discuss in this section is the **Ampere's Law**

$$\oint_c B \cdot dl = \mu_0 I$$



One applicaiton of Ampere's Law is to find the force of magnetic field on any object. Suppose an electric coil generates a magnetic field $B_1(r)$, the force on another electric coil with current I_2 is

$$F_{12} = \int_{C_2} I_2 \overline{dl_2} \times \overline{B_1}$$

Magnetic Dipole

For any **Small** loop carrying a current, there is a dipole similar to electric dipole that characterizes the behavior of the loop.

$$\overline{m} = \text{magnetic dipole moment} = \overline{m} = I \cdot S\overline{a}_n$$

where S stands for the area enclosed by this loop. Using this we can find the magnetic field torque. Suppose there is a small loop in a magnetic field parallel to the loop, we can break the total magnetic field by the loop and the applied field into

$$B = B_\perp + B_\parallel$$

where $B_\perp$ is the self-generated magnetic field, which is found having no net force on the loop(because all the forces are radiallygoing outward), and only the parallel components worth consideration

$$F_\parallel = I\overline{dl} \times \overline{B} = 0$$

but they produce torque along the axis perpendicular to the external field.

$$\boxed{\tau = \overline{m} \times \overline{B_\parallel}}$$

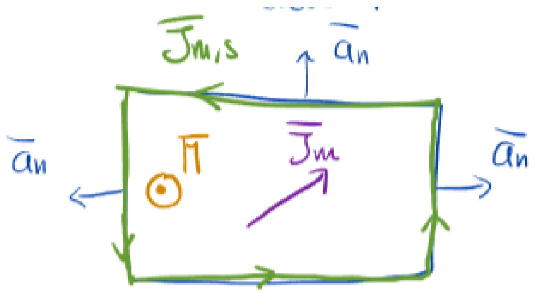
Yet, for material in an external magnetic field, the magnetic dipoles will align with the external magnetic field causing magnetization vector defined as the density of magnetic dipole moment

$$\overline{M} = \frac{\sum \overline{m}_k}{\Delta V}$$

introducing magnetization current densities

$$\begin{cases} \text{Surface Current Density} = J_{m,s} = \overline{M} \times \overline{a}_n \\ \text{Volume current density} = J_m = \nabla \times \overline{M} \end{cases}$$

so through which we can obtain the **Generalized Ampere's Law**



$$\nabla \times B = \mu_0(\overline{J} + \overline{J}_m)$$

$$\nabla \times [\overline{B} - \mu_0\overline{M}] = \mu_0 J$$

such that

$$\overline{H} = \frac{\overline{B}}{\mu_0} - \overline{M} = \text{Magnetic Field [A/m]}$$

Now for different materials go different magnetic dipole moment densities

- Permanent Magnet: $\overline{M} = const$
- Nonlinear Material: $\overline{M} = f(\overline{H})$
- Linear Materials: $\overline{M} = \chi_m\overline{H}$ where χ_m stands for the magnetic susceptibility such that

  $$\overline{B} = \mu_0(1 + \chi_m)\overline{H}$$

  where $1 + \chi_m = \mu_r$ so called **relative Permeability** and $\mu_0(1 + \chi_m) = \mu$ is the absolute permeability of material.

3.1 Inductance

The amount of magnetic flux through a circuit caused by a current carrying circuit is

$$\Phi_{12} = \int_{S_2} B_1 \cdot dS_2 = L_{12} I_1$$

making flux proportional to I_1 making inductance L_{12} the mutual inductance between circuit 1 and circuit 2. The voltage accross an inductor is defined as

$$V_{emf} = -\frac{di}{dt} L$$

3.2 Magnetic Energy

The magnetic Energy is defined as

$$W_m = \frac{1}{2} \int B \cdot \overline{H}$$

and energy stored in an inductor is

$$W_m = \frac{1}{2} L I^2$$

3.3 Faraday's Law

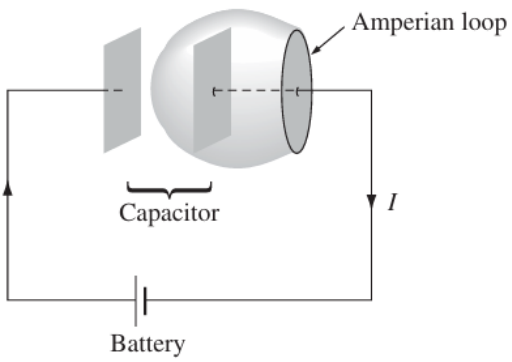
Faraday's law is a simple equation defined by

$$\oint E \cdot dl = -\frac{d}{dt} \int_S \overline{B} \cdot \overline{dS}$$

The LHS is the electromotive force and the RHS is the magnetic flux through S $\Phi(t)$ or in derivative form

$$\nabla \times \overline{E} = -\frac{\partial B}{\partial t}$$

Using the Ampere's Law in this case introduces a contradiction to



$$\oint B \cdot dl = \mu_0 I_{enc}$$

since $I_{enc} = 0$, indicating Ampere's law is only functional steady current. Therefore Maxwell fixed it.
Using the Continuity Equation

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} = -\nabla \cdot (\epsilon_0 \frac{\partial E}{\partial t})$$

integrating this term into the maxwell equation we have

$$\boxed{\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}}$$

Maxwell called this extra term **Displacement Current**

$$J_d = \epsilon_0 \frac{\partial E}{\partial t}$$

3.4 Electromotive Force

To drive a circuit to have current, only having a conservative field is not doing any work. Therefore, a non-conservative field(battery) is introduced such that the total electric field of a circuit can be expressed as

$$E = E_c + E_{NC}$$

This electric field combined to induce a Voltage so called **Electromotive Force (EMF)** defined as

$$\varepsilon = \oint E \cdot dl = \oint E_{NC} \cdot dl = \int_{Start}^{end} E_c \cdot dl$$

which can be understood as the *work done per unit charge* by the battery.

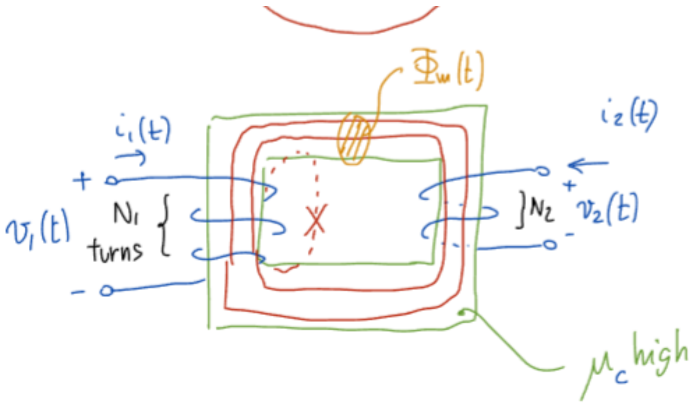
Motional EMF

The total EMF is actually defined by

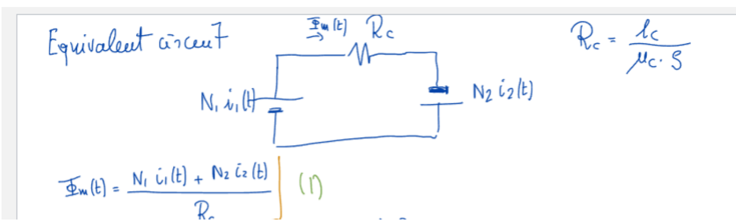
$$V_{emf} = -\frac{d}{dt}\int B \cdot dS + \oint (u \times B) \cdot dl$$

the second term is the motional emf

Transformer



A transformer is used to change voltage from one to another.



$$v_1(t) = L_{11}\frac{di_1}{dt} + \underbrace{L_{12}}_{\mathcal{M}}\frac{di_2}{dt}$$

$$v_2(t) = \underbrace{L_{21}}_{\mathcal{M}}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt}$$

3.5 Maxwell's Equations

Summarizing all the equations

$$\boxed{\begin{array}{lll} \text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho & \text{(Gauss's law)}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0 & \text{(no name)}, \\ \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's law)}, \\ \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{(Ampère's law with Maxwell's correction)}. \end{array}}$$

for Maxwell equations in matters, it reads that

$$\boxed{\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{D} = \rho_f, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}. \end{array}}$$

4 Electromagnetic Wave

4.1 The Wave Equation

The shape of the wave is the same as the shape at time $t = 0$, such that

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

Yet, for a wavefunction f it must follows the **wave equation**

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

- since it involves the square of v, there are two velocities available, where

  $$f(z,t) = h(z + vt)$$

  is another solution

- **Superposition:**
  The general solution to the wave equation is the sume of a wave to the left and to the right

  $$f(z,t) = g(z - vt) + h(z + vt)$$

- **Notation:** The notation for any wave is expressed in complex wave function

  $$\widetilde{f(z,t)} = \tilde{A}e^{i(kz-\omega t)}, \tilde{A} = Ae^{i\delta} = \text{complex amplitude}$$

  with δ being the phase of the wave. Any wave can be expressed as a linear combination of sinusoidal ones

  $$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}dk$$

- **Reflection & Transmittance:** For a transmitted wave, the frequency is the same before and after transmission.

  $$\tilde{f}(z,t) = \begin{cases} \tilde{A}_{incident}e^{i(k_1 z-\omega t)} + \tilde{A}_{reflected}e^{i(-k_1 z-\omega t)}, & \text{for } z < 0, \\ \tilde{A}_{transmitted}e^{i(k_2 z-\omega t)}, & \text{for } z > 0. \end{cases}$$

  Also at the join (z = 0), the displacement to the left must be equal to the right and also the derivative of f must also continuous:

  $$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}, f(0^-,t) = f(0^+,t)$$

  Based on the relationship, we can find that

  $$A_R e^{i\delta_R} = (\frac{v_2 - v_1}{v_2 + v_1})A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = (\frac{2v_2}{v_2 + v_1})A_I e^{i\delta_I}$$

  there are two cases

  - $v_2 > v_1$: $(\delta_R = \delta_T = \delta_I)$

    $$A_R = (\frac{v_2 - v_1}{v_2 + v_1})A_I, \quad A_T = (\frac{2v_2}{v_2 + v_1})A_I$$

  - $v_2 < v_1$: $(\delta_R + \pi = \delta_T = \delta_I)$

    $$A_R = (\frac{v_1 - v_2}{v_2 + v_1})A_I, \quad A_T = (\frac{2v_2}{v_2 + v_1})A_I$$

  - Second medium is infinitely massive

    $$A_R = A_I \quad \text{and} \quad A_T = 0$$

4.2 Polarization

For transverse wave, it has the idea of polarization, it can be divided into two independent states:

$$\tilde{f}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{x}}, \quad \tilde{f}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{y}}, \quad \tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{n}}$$

$\hat{\mathbf{n}}$ defines the plane of vibration, it is called the polarization vector, which is $\hat{n} \cdot \hat{z} = 0$

$$\tilde{f}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\hat{\mathbf{y}}$$

4.3 Electromagnetic Waves in Vacuum

Recall that the maxwell equations in free space are as below

$$\boxed{\begin{array}{lll} \text{(i)} & \nabla \cdot \mathbf{E} = 0 & \text{(Gauss's law)}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0 & \text{(no name)}, \\ \text{(iii)} & \nabla \times \mathbf{E} = -\mu_0 \dfrac{\partial \mathbf{H}}{\partial t} & \text{(Faraday's law)}, \\ \text{(iv)} & \nabla \times \mathbf{B} = \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{(Ampère's law with Maxwell's correction)}. \end{array}}$$

By Faraday's Law, we can find the wave equation for electric field

$$\nabla \times (\nabla \times E) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times H) \rightarrow \boxed{\nabla^2 E = \frac{1}{c_0^2}\frac{\partial^2 E}{\partial t^2}}$$

where $c_0 = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$ = speed of light. Inside a simple dielectric material, the speed of light varies by

$$c = \frac{c_0}{n}$$

where n stands for the **index of refraction** defined as $n = \sqrt{\dfrac{\epsilon}{\epsilon_0}}$. The expression above is the same for magnetic field B.

4.4 Intensity

The flow of energy through electromagnetic waves is described by the **Poynting vector**

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

it points in the direction of energy flow, and the intensity is defined as the magnitude of the time-averaged Poynting vector

$$I = |\langle \mathbf{S} \rangle|$$

4.5 Monochromatic Waves and Complex Notation

- **Monochromatic Wave:**
  it has a harmonic time dependence with a single angular frequency ω. It has the complex form of such by breaking real field into its positive and negative frequency parts:

$$\mathbf{E}(r,t) = \mathbf{E}(r)\frac{e^{-i\phi}}{2}e^{-i\omega t} + \mathbf{E}(r)\frac{e^{i\phi}}{2}e^{i\omega t} = \mathbf{E}^{+}(r)e^{-i\omega t} + \mathbf{E}^{-}(r)e^{i\omega t}$$

  also because the time dependence is of simple form $e^{-i\omega t}$, we can make the identification that

$$\frac{\partial}{\partial t}e^{-i\omega t} = -i\omega e^{-i\omega t} \rightarrow \boxed{\frac{\partial}{\partial t} = -i\omega}$$

Therefore, for a monochromatic field, the Maxwell equations become

$$\nabla \cdot \mathbf{D}^{(+)} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B}^{(+)} = 0 \tag{2}$$

$$\nabla \times \mathbf{E}^{(+)} = i\omega \mathbf{B}^{(+)} \tag{3}$$

$$\nabla \times \mathbf{H}^{(+)} = -i\omega \mathbf{D}^{(+)} \tag{4}$$

Also, the general solution for monochromatic field is

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}^{(+)}(\mathbf{r}, \omega) e^{-i\omega t} \, d\omega$$

- **Intensity** in complex notation, the intensity is similar

$$\begin{aligned} \mathbf{S} &= \mathbf{E} \times \mathbf{H} \\ &= \left(\mathbf{E}^{(+)} e^{-i\omega t} + \mathbf{E}^{(-)} e^{i\omega t}\right) \times \left(\mathbf{H}^{(+)} e^{-i\omega t} + \mathbf{H}^{(-)} e^{i\omega t}\right) \\ &= \mathbf{E}^{(+)} \times \mathbf{H}^{(-)} + \mathbf{E}^{(-)} \times \mathbf{H}^{(+)} + \mathbf{E}^{(+)} \times \mathbf{H}^{(+)} e^{-2i\omega t} + \mathbf{E}^{(-)} \times \mathbf{H}^{(-)} e^{2i\omega t} \\ \rightarrow \langle S \rangle &= \mathbf{E}^{(+)} \times \mathbf{H}^{(-)} + \mathbf{E}^{(-)} \times \mathbf{H}^{(+)} \end{aligned}$$

therefore, by definition, the intensity is equivalent to

$$I = |2Re\{\mathbf{E}^{(+)} \times \mathbf{H}^{(-)}\}|$$

- **Simple Dielectric Media** using the same replacement $\frac{\partial}{\partial t} = -i\omega$ for monochromatic wave in the wave equation for electric field, we can find the **Vector Helmholtz equation**

$$(\nabla^2 + k^2)\mathbf{E} = 0$$

where k $= \omega/c = nk_0$ is the wave number in the medium and $k_0 = \omega/c_0$ is the wave number in vacuum, and n is the refractive index of the material. In a simple dielectric material, due to homogenous property, each component of each field satisfies the equation so E can be a scalar in this case.

- **Plane Wave:** Plane wave is the simpliest solution of the scalar wave equation, which is defined by

$$E^{(+)}(r, t) = E_0^{(+)} e^{i\mathbf{k}\cdot\mathbf{r}} \rightarrow E_0^{(+)} e^{i\mathbf{k}\cdot\mathbf{r} - \omega t} = \text{adding time dependence term}$$

noted that **k** is the wave vector in terms of the components(k_x, k_y, k_z), it satisfies the Helmholtz equation by using the identification $\frac{\partial}{\partial x} = ik_{x_0}$. Also, the phase ϕ of the wave defines that the wavefront is periodic and only dependent on the phase and the number of wavelength q travelled, therefore the **wave fronts(surface of constant phase)** are given by:

$$\mathbf{k} \cdot \mathbf{r} = 2\pi q + \phi$$

where $\mathbf{k} \cdot \mathbf{r}$ are planes perpendicular to **k** separated by the wavelength $\lambda = 2\pi/k$. An easier understanding is that this denotes surfaces that have constant $k \cdot r$

- **Vector Plane Wave** A quick summary, the solution also holds true for magnetic field

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}^{(+)}(\mathbf{r}) e^{-i\omega t} + \text{c.c.} = \mathbf{H}_0^{(+)} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.},$$

By the Maxwell Equations

$$\nabla \times \mathbf{H}^{(+)} = -i\omega\epsilon \mathbf{E}^{(+)} = \mathbf{k} \times \mathbf{H}^{(+)}$$

$$\nabla \times \mathbf{E}^{(+)} = i\omega\mu_0 \mathbf{H}^{(+)} = \mathbf{k} \times \mathbf{E}^{(+)}$$

so here are the findings, **k, H, E** are mutually orthogonal. Where E and H are orthogonal to k so called **Transverse Electromagnetic Waves**. By taking the magnitude of the equations, we require

$$\frac{\omega\epsilon}{k} = \frac{k}{\omega\mu_0} \rightarrow \boxed{k = \frac{n\omega}{c} = nk_0}$$

- **Wave Impedance** The ratio of the electric and magnetic fields

$$\frac{E_0}{H_0} = \frac{1}{n}\sqrt{\frac{\mu_0}{\epsilon_0}} = \eta = \frac{\eta_0}{n}$$

is defined as the **wave impedance of the medium**. Given a certain impedance of the medium, the Poynting vector is going to be

$$\langle \mathbf{S} \rangle = (\frac{|E_0|^2}{\eta} + c.c.)\hat{k} = \frac{E_0^2}{2\eta}\hat{k}$$

And the intensity of the wave will when be

$$I = \frac{E_0^2}{2\eta}$$